PROBLEMS IN LINEAR ALGEBRA (PS01CMTH24)

(1) Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be vanishing at infinity if for given $\epsilon > 0$ there is a compact subset K of X such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$. Let $C_0(X)$ be the collection of all real valued continuous functions on X which are vanishing at infinity. Show that $C_0(X)$ is a vector space over $\mathbb{R}$.

(2) Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to have a compact support if there is a compact subset K of X such that $f(x) = 0$ for all $x \in X \setminus K$. Let $C_c(X)$ be the collection of all real valued continuous functions on X having compact support. Show that $C_c(X)$ is a vector space over $\mathbb{R}$. Also, show that $C_c(X) \subset C_0(X)$.

(3) Let A and B be subspaces of a vector space V. Consider the following subsets of V:
  (a) $A \cap B = \{v \in V : v \in A \text{ and } v \in B\}$.
  (b) $A \cup B = \{v \in V : v \in A \text{ or } v \in B\}$.
  (c) $A + B = \{v \in V : v = a + b,\ a \in A,\ b \in B\}$.
  (c) $A - B = \{v \in V : v = a - b,\ a \in A,\ b \in B\}$.

  For each of the sets (a) to (d) above, either prove that it is a subspace of V or provide a counter example to show that it *need not* be a subspace of V.

(4) If U and W are subspaces of a vector space V then prove that $(U + W)/W$ is isomorphic to $U/(U \cap W)$.

  [Hint:] Use first homomorphism theorem.

(5) Let S and T be two non-empty subsets of V. Then
  (a) $S \subset T$ implies $L(S) \subset L(T)$.
  (b) $L(S \cup T) = L(S) + L(T)$.
  (c) $L(L(S)) = L(S)$.
  (d) S is a subspace of V if and only if $L(S) = S$.

(6) Let $C[0, 1] = \{f : [0, 1] \to \mathbb{C} : f \text{ is continuous}\}$. For some $a \in [0, 1]$, let $M_a = \{f :\in C[0, 1] : f(a) = 0\}$. Show that M_a is a subpsace of $C[0, 1]$. Is the set $\{f \in C[0, 1] : f(0) = \frac{1}{2}\}$ a subspace of $C[0, 1]$? Justify.

(7) Which of the following sets of vectors form a basis for $\mathbb{R}^3$?
  (a) $\{(-1, 0, 0), (1, 1, 1), (1, 2, 3)\}$.
  (b) $\{(0, 1, 2), (1, 1, 1), (1, 2, 3)\}$.
  (c) $\{(-1, 1, 0), (2, 0, 0), (0, 1, 1)\}$.

(8) Write down the basis for the following subspace of $\mathbb{R}^4$:

$$V = \{(x, y, z, t) \in \mathbb{R}^4 : z = x + y,\ x + y + t = 0\}.$$

(9) Let V be a finite dimensional vector space and let A, B and C be subspaces of V. Which of the following statements are true?
  a. $A \cap (B + C) = A \cap B + A \cap C$.
  b. $A \cap (B + C) \subset A \cap B + A \cap C$.

c. $A \cap (B + C) \supset A \cap B + A \cap C$.

If they are true then prove them otherwise give a counter example. (More than one option may be true).

(10) Let $V = \mathbb{R}^4$ be a vector space over $\mathbb{R}$ and M and N be subspaces of V. Find the dimension of $M + N$ in the following cases:
  (a) $M = L(\{(1,1,1,0),(0,-4,1,5)\})$ and $N = L(\{(0,-2,1,2),(1,-1,1,3)\})$.
  (b) $M = L(\{(1,2,0,1),(-1,1,1,1)\})$ and $N = L(\{((0,0,1,1),(2,2,2,2)\})$.

(11) In $V = \mathbb{R}^4$ let W be its subspace given by

$$W = L\{(-1,1,0,0),(1,-1,1,0),(2,-2,3,0)\}).$$

Then find the dimension of V/W.

(12) Let V and W be two finite dimensional vector spaces over F. If T is an isomorphism of V onto W (i.e., $V \cong W$), then prove that T maps a basis of V onto a basis of W.
[**Hint:**] If $\{v_1, \dots, v_n\}$ is a basis of V then show that $\{Tv_1, \dots, Tv_n\}$ is a basis of W.

(13) If V is a finite-dimensional vector space over F then show that any two bases of V have the same number of elements.

(14) Let V be a vector space over a field F and S be a non empty subset of V. Show that $L(S)$ is the smallest subspace of V containing S. Also prove that $L(S)$ is the intersection of all subspaces of V containing S.

(15) Let V be a vector space of all polynomials with real coefficients of degree at most equal to $2n$. Let V_0 stand for the vector space $V_0 = \{p \in V : p(1) + p(-1) = 0\}$ and V_e stand for the vector space which have even degree terms only. Then find $\dim(V_0)$ and $\dim(V_0 \cap V_e)$.

(16) Let V be a vector space such that $\dim(V) = 5$. Let W and Z be subspaces of V such that $\dim(W) = 3$ and $\dim(Z) = 4$. Mention all possible values of $\dim(W \cap Z)$. Justify!

(17) Let V be a real vector space of all polynomials in one variable with real coefficients and of degree lesst than or equal to 5. Let W be the subspace defined by

$$W = \{p \in V : p(1) = p'(2) = 0\}.$$

What is the dimension of W?

(18) Let $W \subset \mathbb{R}^4$ be the subspace defined by

$$W = \{x \in \mathbb{R}^4 : Ax = 0\},$$

where

$$A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}.$$

Write down a basis for W.

(19) Let V and W be vector spaces over F. Then show, with all details, that $\mathrm{Hom}(V, W)$ is a vector space over F.

(20) Find the dual basis corresponding to the following basis of $\mathbb{R}^3$:

$$\{(1,1,0),(1,0,1),(0,1,1)\}.$$

(21) If $S, T \in \mathrm{Hom}(V, W)$ and $S(v_i) = T(v_i)$ for all the elements v_i of a basis of V, then prove that $S = T$.

(22) Let $W = \{a_0 + a_1x + \cdots + a_nx^n : a_0 + a_1 + \cdots + a_n = 0\}$ be a subset of $V = \{p(x) \in F[x] : \deg p(x) \leq n\}$. Show that W is a subspace of V. Find a basis of W.

(23) Let W be a subspace of a vector space V. Show that the annihilator W^0 of W is a subspace of $\widehat{V}$. Also show that if $U \subset W$ then $U^0 \supset W^0$.

(24) If S is a subset of V, then prove that $S^0 = L(S)^0$.

(25) If V is an arbitrary vector space over F and if $T \in A(V)$ is both right-invertible and left-invertible, then prove that the right inverse and left inverse of T must be equal. From this, deduce that the inverse of T is unique.

(26) If U and W are subspaces of a finite dimensional vector space V, then show that:
   (a) $(U + W)^0 = U^0 \cap W^0$.
   (b) $(U \cap W)^0 = U^0 + W^0$.
   [Hint]: Use (1) to deduce (2).

(27) Let V be a vector space over $\mathbb{R}$ of dimension d. Let U be a vector subspace of V and S be a subset of V. State whether the following statements are true or false. Justify!
   (a) If S is a basis of V then $U \cap S$ is a basis of U.
   (b) If $U \cap S$ is a basis of U and $\{s + U \in V/U \mid s \in S\}$ is a basis of V/U then S is a basis of V.
   (c) If S is a basis of U as well as V then $\dim U = d$.

(28) Let V be a finite dimensional vector space over F and $T \in A(V)$. If there exists a $v \neq 0$, $v \in V$ such that $T(v) = 0$ then show that T is singular.

(29) Let $M(n, \mathbb{R})$ be the vector space of $n \times n$ matrices with real entries. Let U be a subset of $M(n, \mathbb{R})$ consisting $\{(a_{ij}) \mid a_{11} + a_{22} + \cdots + a_{nn} = 0\}$. Is it true that U is a vector subspace of $M(n, \mathbb{R})$ over $\mathbb{R}$? If yes, then show that it is a subspace and state its dimension.

(30) If V is finite dimensional and $v_1 \neq v_2$ are in V, prove that there is an $f \in \widehat{V}$ such that $f(v_1) \neq f(v_2)$.

(31) If f and g are in $\widehat{V}$ such that $f(v) = 0$ implies $g(v) = 0$, then prove that $g = \lambda f$ for some $\lambda \in F$.

(32) If F is the field of real numbers, then find W^0 where
   (a) W is spanned by $(1, 2, 3)$ and $(0, 4, -1)$.
   (b) W is spanned by $(0, 0, 1, -1)$, $(2, 1, 1, 0)$ and $(2, 1, 1, -1)$.

(33) Prove that $S \in A(V)$ is regular if and only if whenever $v_1, \ldots, v_n \in V$ are linearly independent, then $S(v_1), S(v_2), \ldots, S(v_n)$ are also linearly independent.

(34) Let W be a subspace of a vector space V. Show that there exists a subspace U of V such that $U \oplus W = V$. Is this U unique? Justify.

(35) If V is an arbitrary vector space over F and if $T \in A(V)$ is right-invertible with *unique* right inverse, then prove that T is invertible.

(36) Prove that the set of all regular elements in $A(V)$ forms a group.

(37) Let V be a finite dimensional vector space over F and $T \in A(V)$. Then $p(x) \in F[x]$ is a minimal polynomial for T if and only if for any other polynomial, say $h(x)$, satisfied by T, we have $p(x) | h(x)$.
   [Hint:] Use division algorithm.

(38) Show that '$\sim$' defined above is an equivalence relation, i.e., being similar is an equivalence relation on $A(V)$.
(39) If $S, T \in A(V)$ are similar then show that they have the same rank. What about the converse?
(40) The element $T \in A(V)$ is called ***nilpotent*** if $T^m = 0$ for some m. If T is nilpotent and if $Tv = \alpha v$ for some $v \neq 0$ in V, $\alpha \in F$, prove that $\alpha = 0$.
This shows that only characteristic root of a nilpotent operator is 0.
(41) Let F be a field and $M_n(F)$ be the collection of all $n \times n$ matrices over F. Show that $M_n(F)$ is an algebra with usual addition of matrices, scalar multiplication and product of matrices.
(42) If $T \in A(V)$ satisfies a polynomial $q(x) \in F[x]$, prove that $S \in A(V)$, S regular, STS^{-1} also satisfies $q(x)$.
(43) If $T \in A(V)$, let $V_0 = \{v \in V \mid T^k v = 0 \text{ for some } k\}$. Prove that V_0 is a subspace and that if $T^m v \in V_0$, then $v \in V_0$.
(44) Let $T \in A(V)$ and $\lambda \in F$ be a characteristic root of T. Define $E_\lambda = \{v \in V \mid Tv = \lambda v\} = \ker(T - \lambda I)$. Show that E_λ is a subspace of V.
(45) Let V be two dimensional vector space over F with basis v_1, v_2. Find the characteristic roots and characteristic vectors for T defined by
  (a) $Tv_1 = v_1 + v_2, Tv_2 = v_1 - v_2$.
  (b) $Tv_1 = 5v_1 + 6v_2, Tv_2 = -7v_2$.
  (c) $Tv_1 = v_1 + 2v_2, Tv_2 = 3v_1 + 6v_2$.
(46) Let $T \in A(V)$. Suppose all the non-zero elements of V are characteristic vectors of T. Then show that T is scalar multiple of the identity map, i.e., that is a $\lambda \in F$ such that $Tv = \lambda v$ for all $v \in V$.
(47) Let V be a finite dimensional vector space over F and $T \in A(V)$. If $\lambda \in F$ is a root of the minimal polynomial for T, then show that λ is a characteristic root of T.
(48) Let V be a finite dimensional vector space over F, $T \in A(V)$ and $m(T)$ be the matrix of T. Show that T and $m(T)$ have the same characteristic roots.
(49) If $T \in A(V)$ is nilpotent and if $\alpha \in F$ such that $\alpha \neq 0$ then prove that $\alpha I + T$ is regular and its inverse is a polynomial in T.
(50) Let $V = \mathbb{R}^3$ and suppose that

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

is the matrix of $T \in A(V)$ in the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Find the matrix of T in the basis:
  (a) $u_1 = (1,1,1)$, $u_2 = (0,1,1)$ and $u_3 = (0,0,1)$.
  (b) $v_1 = (1,1,0)$, $v_2 = (1,2,0)$ and $v_3 = (1,2,1)$.
(51) Prove that, given the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \in M_3(\mathbb{R})$$

then there exists a matrix $C \in M_3(\mathbb{R})$ such that

$$CAC^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(52) If $T \in A(V)$ and if $\lambda \in F$ is a characteristic root of T in F, then let $U_\lambda = \{v \in V : Tv = \lambda v\}$. If $S \in A(V)$ commutes with T, then prove that U_λ is invariant under S.

(53) Give an example of a nilpontent linear transformation on a vector space V.

(54) Let V be a finite dimensional vector space and $T \in A(V)$ is singular. Give an example of an element $S \in A(V)$ such that $TS = 0$ but $ST \neq 0$.

(55) Show that similar linear transformations have the same characteristic roots.

(56) Let $V = F_3[x]$, where F is a field. Define $T \in A(V)$ by

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_0 + \alpha_1(x+1) + \alpha_2(x+1)^2 + \alpha_3(x+1)^3.$$

Compute the matrix of T in the basis

(a) $1, x, x^2, x^3$.

(b) $1, 1+x, 1+x^2, 1+x^3$.

(c) If the matrix in part (a) is A and that in part (b) is B then find a matrix C so that $B = C^{-1}AC$.

(57) If $T \in A(V)$ is nilpotent, prove that T can be brought to triangular form over F, and in that form all the elements on the diagonal are 0.

(58) If $T \in A(V)$ has only 0 as a characteristic root, prove that T is nilpotent.

(59) If S and T are nilpotent linear transformation, which commute, prove that ST and $S+T$ are nilpotent linear transformations.

(60) Define *trace* of a matrix $A = (\alpha_{ij}) \in M_n(F)$ denote it by $\mathrm{tr}(A)$. For any $A, B \in M_n(F)$ and $\lambda \in F$, show that the trace function $\mathrm{tr} : M_n(F) \to F$ is a linear map, i.e.,

(a) $\mathrm{tr}(\lambda A) = \lambda\, \mathrm{tr}(A)$.

(b) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

`[Refer I.N. Herstein, Lemma 6.8.1, Page 314]`

(61) For any $A, B \in M_n(F)$, prove that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

`[Refer I.N. Herstein, Lemma 6.8.1, Page 314]`

(62) Define *transpose* (denote it by A') of a matrix $A = (\alpha_{ij}) \in M_n(F)$. For all $A, B \in M_n(F)$ and $\lambda \in F$, prove that

(a) $(\lambda A)' = \lambda A'$.

(b) $(A+B)' = A' + B'$.

(c) $(A')' = A$.

(d) $(AB)' = B'A'$.

`[Refer I.N. Herstein, Lemma 6.8.5, Page 317]`

(63) Two linear transformation in $A(V)$ having all their characteristic roots in F are similar if and only if they have the same Jordan form (except for the order of their characteristic roots).

(64) Find all possible Jordan forms for 8×8 matrices having $x^2(x-1)^3$ as minimal polynomial.

(65) A matrix $A = (\alpha_{ij})$ is said to be *diagonalizable* if it is similar to a diagonal matrix.

If the multiplicity of each characteristic root of T is 1 and all of them are in F then prove that T is diagonalizable over F.

(66) Show that trace of similar matrices are same.

(67) (a) Let f be a function on $M_n(F)$ having its values in F such that

(a) $f(A+B) = f(A) + f(B)$;

(b) $f(\lambda A) = \lambda f(A)$;

(c) $f(AB) = f(BA)$;

for all $A, B \in M_n(F)$ and all $\lambda \in F$. Prove that there is an element $\alpha_0 \in F$ such that $f(A) = \alpha_0 \operatorname{tr}(A)$ for every $A \in M_(F)$.

(b) If the characteristic of F is 0 and if the f in part (a) satisfies $f(1) = n$, prove that $f(A) = \operatorname{tr}(A)$ for all $A \in M_n(F)$.

(68) If A is invertible, prove that $(A^{-1})' = (A')^{-1}$.

(69) A matrix A is said to be *skew-symmetric matrix* if $A' = -A$. If A is skew-symmetric matrix, prove that the elements on its main diagonal are all 0.

(70) A matrix A is said to be *symmetric matrix* if $A' = A$. If A and B are symmetric matrices, prove that AB is symmetric if and only if $AB = BA$.

(71) Give an example of an A such that $AA' \neq A'A$.