Principles of Mathematics and Biostatistics

PT02CBIC02 (Unit - I)

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Contents

CHAPTER 1

Matrices

1.1 Definition of a Matrix

Definition 1.1.1 (Matrix)**.** A rectangular array of numbers is called a *matrix*.

The horizontal arrays of a matrix are called its *rows* and the vertical arrays are called its *columns*.

Definition 1.1.2 (Order of a matrix)**.** A matrix having m rows and n columns is said to have the order $m \times n$.

A matrix A of order $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n},$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column of the matrix A.

In more concise manner, we also denote the matrix A by $A = (a_{ij})$ thereby suppressing the order of A or $(a_{ij})_{m \times n}$ by specifying its order

A matrix having only one column is called a *column matrix* or simply a *column vector*. Similarly, A matrix having only one row is called a *row matrix* or simply a *row vector*.

Definition 1.1.3 (Equality of two matrices)**.** Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be *equal* if they have the same order $m \times n$ and if $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

1.1.1 Some special matrices

Definition 1.1.4. 1. A matrix in which each entry is zero is called a *zero-matrix*. It is denoted by $\mathbf{0}$. For example,

$$\mathbf{0}_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{0}_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. A matrix having equal number of rows and columns is called a *square* matrix. Thus, order of a square matrix is $n \times n$, represented by n only. For example,
$$A = \begin{bmatrix} 2 & 3 \\ 5 & -4 \end{bmatrix}$$
is a square matrix of order 2.

3. In a square matrix $A = (a_{ij})$ of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called diagonal entries of A and they form principal diagonal of the matrix A.

4. A square matrix $A = (a_{ij})$ of order n in which $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is called the *identity matrix* and is denoted by I_n. For example,
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.2 Operation on matrices

1.2.1 Transpose and Addition of matrices

Definition 1.2.1 (Transpose of a matrix)**.** Let $A = (a_{ij})$ be a matrix of order $m \times n$. The transpose of A is denoted by A' or A^{T} which is a $n \times m$ matrix obtained from A by interchanging its rows and columns.

Thus, if $A = (a_{ij})$ then $A' = (a_{ji})$. For example,
$$\text{If } A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}, \text{ then } A' = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}_{3\times 2}.$$

Theorem 1.2.2. *For any matrix A, we have $(A')' = A$.*

In other words, transpose of transpose of a matrix A is A itself, i.e. taking transpose of A twice gives back the matrix A.

Proof. Let $A = (a_{ij})$. If $A' = (b_{ij})$ then by definition of transpose, $b_{ij} = a_{ji}$. Now, if $(A')' = (c_{ij})$, then
$$c_{ij} = b_{ji} = a_{ij}.$$
Thus, $(A')' = A$. □

Definition 1.2.3 (Addition of matrices)**.** Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of order $m \times n$. Then the sum of A and B is denoted by $A + B$ is the matrix $C = (c_{ij})$, where c_{ij} is given by
$$c_{ij} = a_{ij} + b_{ij}.$$
Thus, the addition of two matrices is defined as entry-wise.

Note: Note that, we can define the sum of two matrices only if the order of the two matrices are same.

Example 1.2.4. 1. $\begin{bmatrix} 1 & 3 \end{bmatrix} + \begin{bmatrix} -10 & 4 \end{bmatrix} = \begin{bmatrix} -9 & 7 \end{bmatrix}$.

2. $\begin{pmatrix} 2 & 8 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 2 & 3 \end{pmatrix}$.

Definition 1.2.5 (Multiplication by a scalar)**.** Let $A = (a_{ij})$ be a $m \times n$ matrix. Then for any real number $k \in \mathbb{R}$, we define $kA = (ka_{ij})$. For example, for $k = 5$, if

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}, \text{ then } 5A = \begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}.$$

1.2.2 Multiplication of matrices

Definition 1.2.6. Let $A = (a_{ij})$ be a $m \times n$ matrix and $B = (b_{ij})$ be a $n \times r$ matrix. Then the product AB is a matrix $C = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Note: Observe that the matrix multiplication AB is defined if only if

Number of Columns of A = Number of Rows of B

Similarly, the matrix multiplication BA is defined if only if

Number of Columns of B = Number of Rows of A

Example 1.2.7. For example, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}.$$

Here, the matrix A is of order 2×3 and B is a 3×3 matrix. Hence, the matrix product AB is defined and AB will be a 2×3 matrix. However, the matrix product BA is not possible as the number of columns of B are 3 and number of rows of A are 2, not being same. Now,

$$AB = \begin{bmatrix} 1+0+2 & 2+0+0 & 1+6+12 \\ 2+0+1 & 4+0+0 & 2+12+4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 19 \\ 3 & 4 & 18 \end{bmatrix}.$$

Definition 1.2.8. Two square matrices A and B are said to *commute* if $AB = BA$.

Remark 1.2.9. 1. If A is a square matrix of order n, then $I_nA = A = AI_n$, where I_n is the $n \times n$ identity matrix defined earlier.

2. In general matrix multiplication is not commutative, i.e. $AB = BA$ is **not true**, in general, for all square matrices A and B. For example, consider the two 2×2 matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Check that the matrix product

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

1.3 Determinant

Notation: For a $n \times n$ matrix A, we denote the submatrix of A obtained by deleting i^{th} row and j^{th} column by $A(i|j)$.

Example 1.3.1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$. Then

$$A(1|2) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},$$

whereas,

$$A(2|2) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

Definition 1.3.2 (Determinant of a square matrix)**.** Let A be a square matrix of order $n \times n$. With A we associate a number, called the determinant of A, denoted by $\det(A)$ or $|A|$ and defined inductively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A(1|j)).$$

Note that determinant can be defined for any i instead of 1, i.e. for any fixed row i, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)).$$

Notation: For a square matrix A of order n, let $C_{ij} = \det(A(i|j))$, that is, C_{ij} is the determinant of the submatrix of A obtained by deleting the i^{th} row and j^{th} column.

Example 1.3.3.

1. For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, its determinant is given by

$$\det(A) \text{ or } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}C_{11} - a_{12}C_{12}.$$

2. For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, its determinant is given by

$$\begin{aligned} \det(A) \text{ or } |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}). \end{aligned}$$

Example 1.3.4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Then determinant of A is given by

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 1(6-2) - 2(4-1) + 3(4-3) \\ &= 4 - 6 + 3 = 1. \end{aligned}$$

Notation: $A_{ij} = (-1)^{i+j} C_{ij} = (-1)^{i+j} \det(A(i|j))$.

Thus, A_{ij} is $(-1)^{i+j}$ times the determinant of the submatrix of A obtained by deleting the i^{th} row and j^{th} column from A.

1.4 Inverse of a matrix

1.4.1 Adjoint of a matrix

Definition 1.4.1 (Adjoint of a matrix)**.** Let A be a $n \times n$ matrix. Then the adjoint of A is an $n \times n$ matrix denoted by $Adj(A)$ and defined as the transpose of the matrix (A_{ij}), i.e. $Adj(A) = (A_{ji})$.

Thus, for a 3×3 matrix $A = (a_{ij})$, $Adj(A)$ is the transpose of the following matrix:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Example 1.4.2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Then to find $Adj(A)$ first we compute the following matrix.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Now, $A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6 - 2 = 4$, $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -1(4-1) = -3$, $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$, $A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -1(4-6) = -1(-2) = 2$ and so on ....

Then the matrix $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$. Hence, the adjoint of A is the transpose of this matrix and is given by

$$Adj(A) = \begin{bmatrix} 4 & 2 & -7 \\ -3 & -1 & 5 \\ 1 & 0 & -1 \end{bmatrix}.$$

1.4.2 Inverse of a matrix

Definition 1.4.3 (Inverse of a matrix)**.** Let A be a $n \times n$ matrix. If $\det(A) \neq 0$ then inverse of A is defined. It is denote by A^{-1} and defined as

$$A^{-1} = \frac{1}{\det(A)} Adj(A) \text{ or } \frac{Adj(A)}{|A|}.$$

Remark 1.4.4. If A is an $n \times n$ matrix which is invertible, then

$$AA^{-1} = I_n = A^{-1}A,$$

where I_n is the identity matrix of order n.

Example 1.4.5. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. Check whether it is possible to find the inverse of A. If yes, then compute A^{-1}.

Solution. (Exercise) Check that $\det(A) = -2$ which is non-zero. Hence, A^{-1} exists, or in other words, matrix A is invertible. Also, compute the adjoint of A. Check that

$$Adj(A) = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix}.$$

Then by above formula, check that,

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

1.5 Exercises

Exercise 1.5.1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Do the following:

1. Compute $A + B$.
2. Find A', B' and C', i.e. compute the transposes of matrices A, B and C.
3. Compute $4A - 2B$.
4. $C + C'$ is defined. (`True` or `False`). If True, then find $C + C'$. If False, then state the reason.
5. $A + C$ is defined. (`True` or `False`). If True, then find $A + C$. If False, then state the reason.
6. Show that A and B do not commute. That is, compute AB and BA and show that $AB \neq BA$.

Exercise 1.5.2. Let $A = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$. Compute the following product of matrices, if possible:

1. $CB, A'B$ and AB.
2. $B'CA$ and BCA.

Exercise 1.5.3. Let $A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$. Compute $Adj(A)$ and hence show that

$$A \cdot Adj(A) = 11\, I_3.$$

Exercise 1.5.4. Find determinants of the following matrices.

$$A = \begin{pmatrix} 4 & -2 \\ 1 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Exercise 1.5.5. Check in the following cases, whether the matrices are invertible or not. If yes, then compute their inverse.

1. $A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
3. $C = \begin{pmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Exercise 1.5.6. Show that $\begin{pmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{pmatrix}$ is the inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & -3 \end{pmatrix}$.

CHAPTER 2

Graph Theory

2.1 Introduction

2.1.1 Graph and its components

Definition 2.1.1 (Graph)**.** A graph $G = (V, E)$ consists of a set of objects $\{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is associated with an unordered pair of vertices (v_i, v_j).

The vertices v_i and v_j associated with edge e_k are called its *end vertices*.

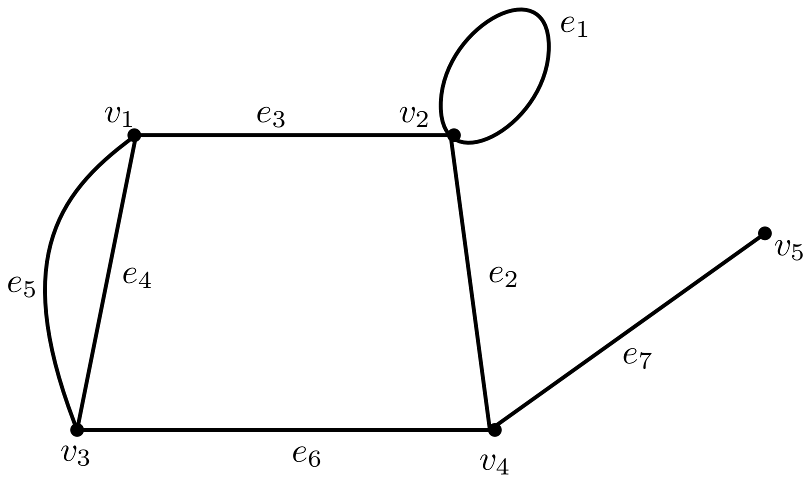


Figure 2.1: Graph with five vertices and seven edges

Definition 2.1.2 (Self-loop)**.** An edge having the same vertex as both its end-vertices is called a *self-loop*. Thus, an edge associated with the pair of vertices (v_i, v_i) is called a self-loop.

In Figure 2.1, the edge e_1 is a self-loop having both its end-vertices as v_2.

Definition 2.1.3 (Parallel edges)**.** If two edges have the same pair of end-vertices then they are called *parallel edges*.

For example, in Figure 2.1, the edge e_4 and e_5 are parallel edges having end-vertices as v_1 and v_3.

Definition 2.1.4 (Simple graph)**.** A graph that has neither parallel edges nor self loops is called a *simple graph*.

2.1.2 Incidence and Degree

Definition 2.1.5. If a vertex v_i is an end vertex of some edge e_j, we say that v_i is incident on e_j or e_j is incident on v_i, i.e. v_i and e_j are incident on each other.

Definition 2.1.6 (Adjacent edges)**.** Two non-parallel edges are said to be *adjacent* if they are incident on a common vertex.

Definition 2.1.7 (Adjacent vertices)**.** Two vertices are said to be *adjacent* if they are the end vertices of the same edge.

Definition 2.1.8 (Degree of a vertex)**.** The number of edges incident on a vertex v_i with self-loops counted twice, is called the *degree* of the vertex v_i. It is denoted by $d(v_i)$.

For example, in Figure 2.1, $d(v_1) = d(v_3) = d(v_4) = 3$, $d(v_2) = 4$ and $d(v_5) = 1$.

Remark 2.1.9. Let G be a graph with e edges and n vertices, $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all the vertices in G is twice the number of edges in G, i.e. $2e$. Thus,

$$\sum_{i=1}^{n} d(v_i) = 2e. \tag{2.1}$$

Theorem 2.1.10. *The number of vertices of odd degree in a graph is always even.*

Proof. Let us consider the vertices with odd and even degrees separately. Then the sum in left hand side of equation 2.1 can be written as a sum of two sums, where each sum is taken over the vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_i) + \sum_{\text{odd}} d(v_i). \tag{2.2}$$

The left hand side of above equation is $2e$ which is even number. The first sum on right hand side of above equation is also even (being the sum of all even numbers). Hence, the second sum in right hand side of above equation must also be even:

$$\sum_{\text{odd}} d(v_i) = \text{an even number.}$$

□

2.1.3 Isolated vertex, Pendant vertex, Null Graph

Definition 2.1.11 (Isolated vertex)**.** A vertex having no incident edge is called an *isolated vertex*. In other words, a vertex with 0 degree is called an isolated vertex.

Definition 2.1.12 (Pendant vertex)**.** A vertex of degree one is called a *pendant vertex*.
For example, in Figure 2.1, v_5 is a pendant vertex as $d(v_5) = 1$.

Definition 2.1.13 (Null graph)**.** A graph without edges is called a *null graph*. That is, a graph $G = (V, E)$ in which the set E is empty is called a null graph.
Note that, in null graph every vertex is isolated.

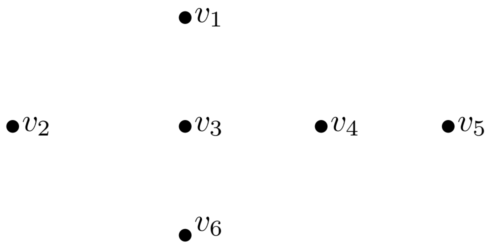
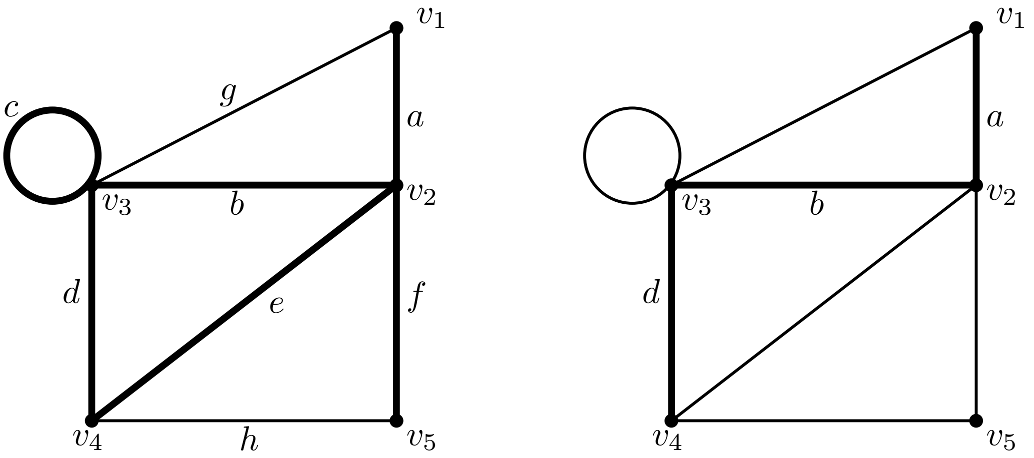


Figure 2.2: Null graph with six vertices

2.2 Paths and Circuits

2.2.1 Walks, Paths and Circuits



(a) An Open Walk (b) A Path of Length Three

Figure 2.3: A walk and a path

Definition 2.2.1 (Walk)**.** A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it and no edge appears (or is covered) more than once.

A vertex, however, may appear more than once in a walk. For example, in Figure 2.3(a),

$$v_1 \ a \ v_2 \ b \ v_3 \ c \ v_3 \ d \ v_4 \ e \ v_2 \ f \ v_5$$

is a walk shown with heavy (dark) lines. A walk is also called *edge train* or a *chain*.

Vertices with which a walk begins and ends are called its *terminal vertices*. Here, v_1 and v_5 are the terminal vertices for the walk shown above.

Definition 2.2.2 (Closed walk)**.** A walk that begins and ends at the same vertex is called a *closed walk*. A walk which is not closed is called an *open walk*.

Definition 2.2.3 (Path)**.** An open walk in which no vertex appears more than once is called a *path*.

For example, in Figure 2.3,

$$v_1 \ a \ v_2 \ b \ v_3 \ d \ v_4$$

is a path, whereas

$$v_1 \ a \ v_2 \ b \ v_3 \ d \ v_4 \ e \ v_2 \ f \ v_5$$

is an open walk but not a path (as v_2 is repeated in the walk). In other words, a path does not intersect itself.

Definition 2.2.4 (Length of a path)**.** The number of edges appearing in a path is called the *length of the path*. The path given in the above example has length 3.

Thus, it follows that, a single edge which is not a self-loop is a path of length 1.

The terminal vertices of a path are of degree one, whereas the rest of the vertices called the *intermediate vertices* are of degree two. This degree is counted only with respect to the edges included in the path and not the entire graph in which the path is contained.

Definition 2.2.5 (Circuit or Cycle)**.** A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called the *circuit*. That is, a circuit is a closed and non-intersecting walk. A circuit sometimes is also called a *cycle*.

For example, in Figure 2.3, $v_2 \ b \ v_3 \ d \ v_4 \ e \ v_2$ is a circuit. Clearly, every vertex is a circuit is of degree 2.

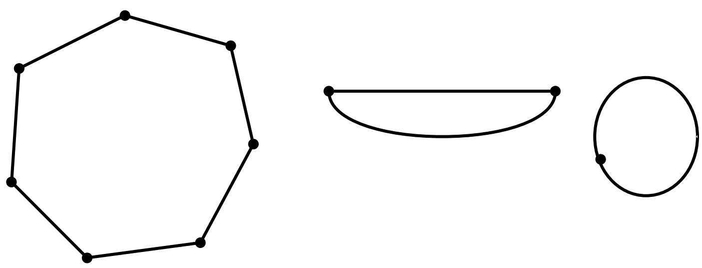


Figure 2.4: Three different circuits

2.2.2 Connected Graphs, Disconnected Graphs and Components

Informally speaking, a graph is connected if we can reach any vertex from any other vertex by traveling along the edges. More precisely:

Definition 2.2.6 (Connected and disconnected graph)**.** A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. A graph which is not connected is called a *disconnected* graph.

For example, the graphs figures 2.1 and 2.3 are connected, where the one in Figure 2.5 is a disconnected graph.

Clearly, a null graph with more than one vertex is a disconnected graph.

Definition 2.2.7 (Components)**.** A disconnected graph consists of two or more connected graphs. Each of this connected subgraphs or connected parts is called a *component*.

Figure 2.5 is a disconnected graph with two components.

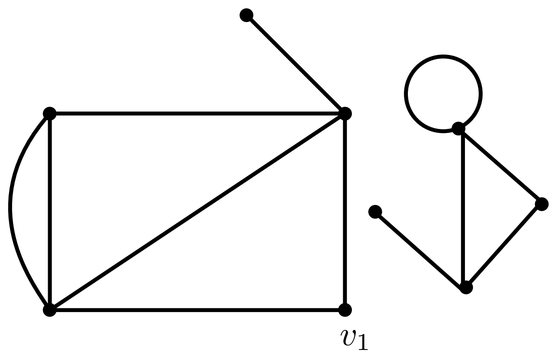


Figure 2.5: A disconnected graph with two components

Theorem 2.2.8. *A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and other in V_2.*

Theorem 2.2.9. *If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.*

2.2.3 Euler Graphs

Definition 2.2.10 (Euler Graph)**.** A closed walk which passes through (or covering) every edge of a graph G exactly once is called an *Euler line*. In other words, a closed walk containing all the edges of a graph is called an Euler line.

A graph that consists of an Euler line is called an *Euler graph*.

Note that vertex repetition is allowed in an Euler line. An Euler graph is always connected, except if the graph has some isolated vertices. The following result, enables us to tell immediately that whether a given graph is an Euler graph or not:

Theorem 2.2.11. *A given connected graph G is an Euler graph if and only if all the vertices of G are of even degree.*

Theorem 2.2.12. *A connected graph G is an Euler graph if and only if it can be decomposed into circuits.*

2.2.4 Hamiltonian Paths and Circuits

Definition 2.2.13 (Hamiltonian circuit)**.** A closed walk in a connected graph G that traverses every vertex of G vertex exactly once, except the starting vertex at which walk also terminates, is called a *Hamiltonian circuit* or a *Hamiltonian cycle.*

In other words, a circuit in a connected graph G is said to be *Hamiltonian* if it includes every vertex of G. Hence, a Hamiltonian circuit in a graph with n vertices contains n edges.

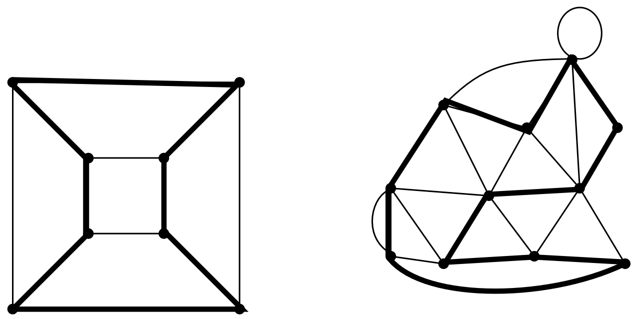


Figure 2.6: Hamiltonian circuits

Definition 2.2.14 (Hamiltonian path)**.** If we remove any one edge from a Hamiltonian circuit, we are left with a path which is called a *Hamiltonian path.* Clearly, a Hamiltonian path in a graph G traverses every vertex of G.

Definition 2.2.15 (Complete Graph)**.** A simple graph in which there exists an edge between every pair of vertices is called a *complete graph.* Complete graphs with two, three and four vertices are shown in the following figure 2.7.

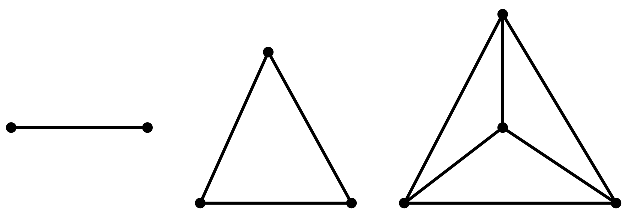


Figure 2.7: Compete graphs with two, three and four vertices

Since every vertex is joined with every other vertex through one edge, the degree of every vertex is $n-1$ in a complete graph G with n vertices. Hence, the total number of edges in G is $\dfrac{n(n-1)}{2}$.

2.3 Trees and Fundamental Circuits

2.3.1 Trees and its properties

Definition 2.3.1 (Tree)**.** A *tree* is a connected graph without any circuits.

A tree is usually denoted by T. Trees with one, two, three and four vertices are shown in the figure 2.8 below.

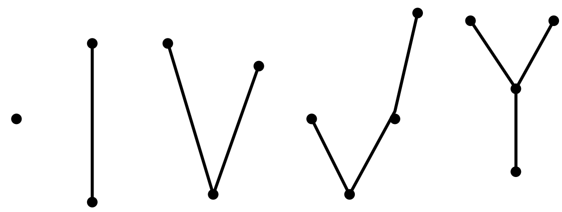


Figure 2.8: Trees with one, two, three and four vertices

Some properties of trees:

Theorem 2.3.2. *There is one and only one path between every pair of vertices in a tree, T.*

Proof. Since T is a connected graph, there must exists at least one path between every pair of vertices in T. Now suppose that between two vertices a and b in T there are two distinct paths. The union of these two paths will contain a circuit and so T cannot be a tree. □

Theorem 2.3.3. *If in a graph G there is one and only one path between every pair of vertices then G is a tree*

Proof. Existence of a path between every pair of vertices assures that G is a connected graph. A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a and b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G cannot have any circuit. Hence, G is a tree. □

Theorem 2.3.4. *A tree with n vertices has $n-1$ degrees.*

Theorem 2.3.5. *Any connected graph with n vertices and $n-1$ edges is a tree.*

Definition 2.3.6 (Minimally connected)**.** A connected graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph.

Theorem 2.3.7. *A graph is a tree if and only if it is minimally connected.*

Proof. A minimally connected graph cannot have a circuit otherwise, we could remove one of its edges in the circuit and still leave the graph connected. Thus, a minimally connected graph is a tree.

Conversely, suppose a graph G is a tree. if a connected graph G is not minimally connected then there must exists an edge e_i in G such that $G - e_i$ is (still) connected. Therefore, e_i is in some circuit, which implies that G is not a tree. Hence, a tree is minimally connected. □

Theorem 2.3.8. *In any tree (with two or more vertices), there are at least two pendant vertices.*

2.3.2 Distance and Centers in a Tree

Definition 2.3.9 (Distance between two vertices)**.** In a connected graph G, the distance $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path between them, i.e. the number of edges in the shortest path between them.

In a tree, since there is only one path between every pair of vertices, the determination of distance between two vertices in a tree becomes easier, i.e. it is the length of the (number of edges in) path between them.

Definition 2.3.10 (Eccentricity and center)**.** The *eccentricity* $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G, that is

$$E(v) = \max_{v_i \in G} d(v, v_i).$$

A vertex with minimum eccentricity in graph G is called a *center* of G.

Theorem 2.3.11. *Every tree has either one or two centers.*

2.3.3 Spanning Trees

Definition 2.3.12 (Spanning tree)**.** A tree T is said to be a *spanning tree* of a connected graph G if T is a subgraph of G and T contains all the vertices of G.

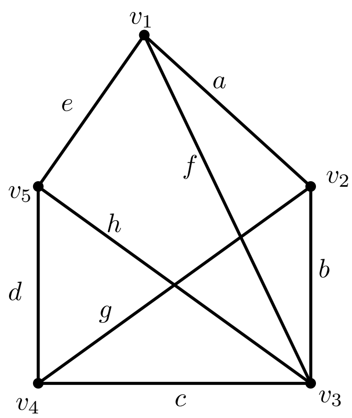


Figure 2.9: Graph with 5 vertices and 8 edges

Consider the graph above (Figure 2.9) with 5 vertices and 8 edges. Some of the spanning trees (not all) of the above graph are listed below.

- $\{a, b, c, d\}$
- $\{b, f, e, d\}$
- $\{g, c, f, e\}$
- $\{h, f, a, g\}$
- $\{g, b, f, e\}$
- $\{f, b, g, d\}$

Theorem 2.3.13. *Every connected graph has at least one spanning tree.*

Proof. Let G be a connected graph. If G has no circuits, it is its own spanning tree. If G has a circuit, delete an edge from the circuit. This will still leave the graph connected. If there are more circuits, repeat the procedure till an edge from the last circuit is deleted. This leaves a connected graph without any circuits that contains all the vertices of G. Thus, every connected graph has at least one spanning tree. □

Definition 2.3.14 (Branches and chords)**.** An edge in a spanning tree T is called a *branch* of T. An edge of graph G which is not in the spanning tree T is called a *chord*.

2.3.4 Fundamental Circuits

Theorem 2.3.15. *A connected graph G is a tree if and only if adding an edge between two vertices in G creates exactly one circuit.*

Proof. We know that, there is one and only one path between every pair of vertices in a tree. Thus, adding an edge adds an additional path and hence a circuit. □

Definition 2.3.16 (Fundamental circuit)**.** Adding any chord (i.e. an edge not in a spanning tree) to the spanning tree T will create exactly one circuit. Such a circuit formed by adding a chord is called a *fundamental circuit*.

In the above graph given in Figure 2.9, consider the spanning tree $T = \{d, e, a, b\}$. Note that the spanning tree has 4 edges which are branches and the remaining 4 edges in the graph are chords. Thus, these 4 chords will give 4 fundamental circuits with respect to T. They are $\{a, b, c, d, e\}$, $\{a, b, f\}$, $\{b, c, g\}$, $\{a, b, h, e\}$ obtained by adding chords c, f, g and h respectively.

Exercise 2.3.17. Find fundamental circuits with respect to other spanning trees of the graph 2.9.

2.4 Matrix representation of Graphs

2.4.1 Incidence matrix

Let G be a graph with n vertices, e edges and **no self-loops**. Define an $n \times e$ (n by e) matrix $A = (a_{ij})$, whose n rows correspond to the n vertices and the e columns correspond to the e edges as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertx } v_i \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Such a matrix A is called the *vertx edge incidence matrix* or simply *incidence matrix*. Consider the graph given below. We find the incidence matrix of the same.

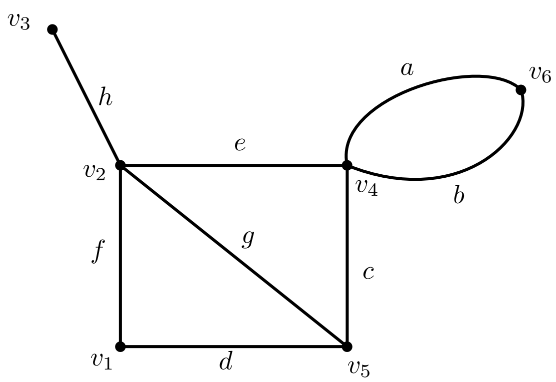


Figure 2.10: Graph with 6 vertices and 8 edges

The incidence matrix A of the above graph is a 6×8 matrix given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

One can clearly make the following observations about the incidence matrix A:

1. Since every edge is incident on exactly two vertices (as there are no self-loops), each column of A
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's represents an isolated vertex.
4. Parallel edges in a graph produce identical columns. In other words, two columns in the incidence matrix are identical if the two edges represented by them are parallel.
5. If a graph G is disconnected with components g_1 and g_2 then the incidence matrix $A(G)$ of G can be written in the following form:

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array} \right].$$

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

2.4.2 Circuit Matrix

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $B = (b_{ij})$ of G is a $q \times e$ binary matrix defined as follows:

$$b_{ij} = \begin{cases} 1, & \text{if } i\text{th circuit includes } j\text{th edge and} \\ 0, & \text{otherwise.} \end{cases}$$

The graph in the above Figure 2.10 has four different circuits, $\{a, b\}$, $\{c, e, g\}$, $\{d, f, g\}$ and $\{c, d, f, e\}$. Therefore, its circuit matrix is a 4×8 binary matrix and is given by

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The following observations can be made about a circuit matrix $B(G)$ of a graph G:

1. A column of all zeros corresponds to a non-circuit edge, i.e. an edge that does not belong to any circuit.
2. Each row of $B(G)$ is a circuit vector.
3. Unlike incidence matrix, a circuit matrix is capable of representing a self-loop and the corresponding row will have a single 1.

4. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
5. If graph G is disconnected with components g_1 and g_2 then the circuit matrix $B(G)$ can be written in a block-diagonal form as follows:

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right],$$

where $B(g_1$ and $B(g_2)$ are circuit matrices of g_1 and g_2.
6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.

2.4.3 Adjacency matrix

The *adjacency matrix* of a graph G with n vertices and **no parallel edges** is an n by n symmetric binary matrix $X = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0, & \text{if there is no edge between them.} \end{cases}$$

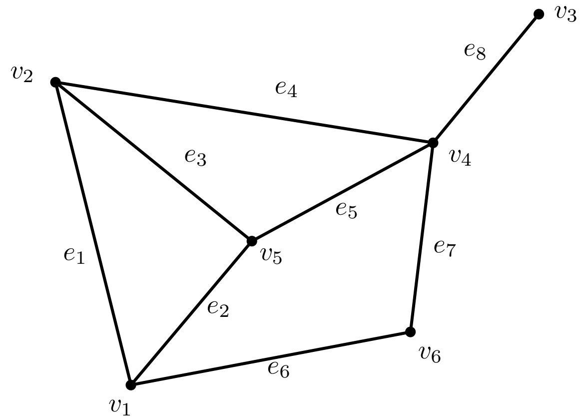


Figure 2.11: A graph with 6 vertices

The adjacency matrix of the graph given in Figure 2.11 is a 6×6 matrix given as follows:

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Observations about the adjacency matrix that can be made immediately are:
1. The entries along the principal (main) diagonal of X are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision of parallel edges.

3. If the graph has no self-loops (and of course no parallel edges), the degree of a vertex equals the number of 1's in the corresponding row or column of X.
4. Permutation of row **and** of the corresponding columns imply reordering the vertices. It must be noted that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged.
5. A graph G is disconnected with components g_1 and g_2 if and only if its adjacency matrix $X(G)$ can be partitioned as follows:

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right],$$

where $X(g_1)$ and $X(g_2)$ are the adjacency matrices of the components g_1 and g_2 respectively.
6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

2.5 Directed Graphs or Digraphs

Definition 2.5.1. A *directed graph* or *digraph* G consists of a set $V = \{v_1, v_2, \ldots\}$ of vertices, a set $E = \{e_1, e_2, \ldots\}$ of edges and a mapping that maps every edge onto some *ordered* pair of vertices (v_i, v_j).

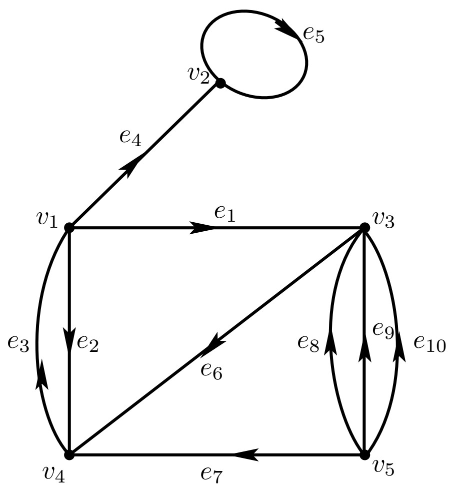


Figure 2.12: Directed graph with 5 vertices and 10 edges

In a digraph, an edge is not only incident on a vertex but also *incident out of* a vertex and *incident into* a vertex. If edge e_k is incident out of a vertex v_i and incident into a vertex v_j, then v_i is called the initial vertex and v_j is called the terminal vertex of e_k. For example, in the above graph (Figure 2.12) v_5 is the initial vertex and v_4 is the terminal vertex of the edge e_7.

An edge for which initial and terminal vertices are same is called a *self-loop*. For example, e_5 in the above graph. The number of edges incident out of a vertex v_i is called the *out-degree* of v_i and is denoted by $d^+(v_i)$. The number of edges incident into a vertex v_i is called the *in-degree* of v_i and is denoted by $d^-(v_i)$. For example,

$$d^+(v_1) = 3, \quad d^-(v_1) = 1,$$
$$d^+(v_2) = 1, \quad d^-(v_2) = 2,$$
$$d^+(v_5) = 4, \quad d^-(v_5) = 0.$$

Clearly, in a digraph G with n vertices and e edges, the sum of all in-degrees is equal to the sum of all out-degrees and each sum is equal to the number of edges in G, that is,

$$\sum_{i=1}^{n} d^+(v_i) = e = \sum_{i=1}^{n} d^-(v_i).$$

A vertex for which in-degree and out-degree are both equal to zero is called an *isolated* vertex. A vertex v in a digraph is called a *pendant* vertex if it is of degree (in-degree or out-degree) one, that is, if

$$d^+(v) + d^-(v) = 1.$$

Two directed edges are said to be *parallel* if they are mapped onto the same ordered pair of vertices. For example in Figure 2.12, e_8, e_9 and e_{10} are parallel edges whereas e_2 and e_3 are not parallel.

2.5.1 Some types of Digraphs

Simple Digraphs A digraph that has no self-loop or parallel edges is called a simple digraph.

Asymmetric Digraph Digraphs that have at most one directed (either no edge or one edge) edge between a pair of vertices, but are allowed to have self-loops, are called *asymmetric* or *antisymmetric*.

Symmetric Digraph Digraphs in which for every edge (a, b) (i.e. from vertex a to b) there is also and edge (b, a) (i.e. from vertex b to a).

A digraph that is both simple and asymmetric is called a *simple asymmetric digraph*. Similarly, a digraph which is both simple and symmetric is called a *simple symmetric digraph*.

Complete Digraphs A *complete symmetric digraph* is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.

A *complete asymmetric digraph* is an asymmetric digraph in which there is exactly one edge between every pair of vertices. The following are the examples:

Complete symmetric digraph of 4 vertices

Complete asymmetric digraph with 4 vertices

Figure 2.13: Directed graph with 5 vertices and 10 edges

A complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$ edges, but a complete symmetric digraph of n vertices contains $n(n-1)$ edges.

Balanced Digraph A digraph is said to be *balanced* if for every vertex v_i the in-degree equals the out-degree, that is, $d^+(v_i) = d^-(v_i)$.

2.5.2 Directed Paths and Connectedness

A *directed walk* from a vertex v_i to v_j is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j, such that each edge is directed from the vertex preceding it to the vertex following it. A *semiwalk* in a directed graph is a walk in the corresponding undirected graph, but is not a directed walk. A *walk* in a digraph can mean either a directed walk or a semiwalk.

Similarly, the definitions of path, directed path, semipath, circuit, directed circuit, semicircuit can be given in a digraph. For example, in Figure 2.12, the alternating sequence

$$v_5\ e_8\ v_3\ e_6\ v_4\ e_3\ v_1$$

is a directed path from v_5 to v_1, where as

$$v_5\ e_7\ v_4\ e_6\ v_3\ e_1\ v_1$$

is a semipath (being a path in the corresponding undirected graph) but not a directed path. Again in the same graph, the set of edges $\{e_1, e_6, e_3\}$ is a directed circuit but $\{e_1, e_6, e_2\}$ is a semicircuit. Both of them are circuits.

Connected Digraphs Recall that, an undirected graph was defined to be connected if there was at least one path between every pair of vertices. In the case of directed graphs, there are two different types of paths. Consequently, there are two different types of connectedness in the case of digraphs.

A digraph G is said to be *strongly connected* if there is at least one directed path from every vertex to every other vertex. A digraph G is said to be *weakly connected* if its corresponding undirected graph is connected but G is not strongly connected. A digraph that is not connected (strongly or weakly) is said to be disconnected.

Euler Digraphs In a digraph G a closed directed walk which traverses (or covers) every edge of G exactly once is called a *directed Euler line.* A digraph containing a directed Euler line is called an *Euler digraph.* The following graph is an Euler digraph, in which the walk $a\ b\ c\ d\ e\ f$ is an Euler line.

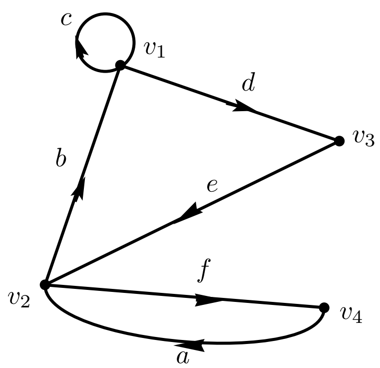


Figure 2.14: Euler digraph

Definition 2.5.2 (Tree with directed edges)**.** A *tree* is a connected digraph without any circuits - i.e. neither a directed circuit nor a semicircuit.

2.5.3 Matrices of digraphs

1. *Incidence Matrix*: The incidence matrix of a digraph with n vertices, e edges and no self-loops is an $n \times e$ matrix $A = (a_{ij})$ whose rows correspond to vertices and columns corresponds to edges, such that

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge is incident out } i\text{th vertex and} \\ -1, & \text{if } j\text{th edge is incident into } i\text{th vertex and} \\ 0, & \text{if } j\text{th edge is not incident on } i\text{th vertex.} \end{cases}$$

2. *Circuit Matrix*: Let G be a digraph with e edges and q circuits (directed circuits or semicircuits). An arbitrary orientation (clockwise or counterclockwise) is assigned to each of the q circuits. Then a circuit matrix $B = (b_{ij})$ of the digraph G is a $q \times e$ matrix defined as

$$b_{ij} = \begin{cases} 1, & \text{if } i\text{th circuit includes } j\text{th edge, and the orientation of the edge and circuit coincide} \\ -1, & \text{if } i\text{th circuit includes } j\text{th edge, but the orientation of the two are opposite,} \\ 0, & \text{if } i\text{th circuit does not include } j\text{th edge.} \end{cases}$$

3. *Adjacency Matrix*: Let G be a digraph with n vertices, containing no parallel edges. Then the adjacency matrix $X = (x_{ij})$ of the digraph G is an $n \times n$ binary matrix whose elements (or entries) are defined as

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } i\text{th vertex to } j\text{th vertex,} \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 2.5.3. Find the incidence matrix, circuit matrix and the adjacency matrix in the following digraph.

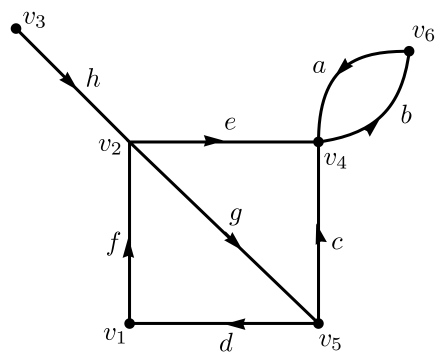


Figure 2.15: Digraphs and its matrices